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MATHEMATICS
HIGHER LEVEL
PAPER 1

Candidate session number

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Thursday 9 May 2013 (afternoon)

Examination code

2	2	1	3	–	7	2	0	3

2 hours

INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions in the boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the ***Mathematics HL and Further Mathematics SL*** *information booklet* is required for this paper.
- The maximum mark for this examination paper is *[120 marks]*.

0116

14 pages

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

SECTION A

Answer ***all*** *questions in the boxes provided. Working may be continued below the lines if necessary.*

1. *[Maximum mark: 6]*

(a) If $w = 2 + 2\mathrm{i}$, find the modulus and argument of w. *[2 marks]*

(b) Given $z = \cos\left(\frac{5\pi}{6}\right) + \mathrm{i}\sin\left(\frac{5\pi}{6}\right)$, find in its simplest form $w^4 z^6$. *[4 marks]*

2. *[Maximum mark: 6]*

Consider the points $A(1, 2, 3)$, $B(1, 0, 5)$ and $C(2, -1, 4)$.

(a) Find $\vec{AB} \times \vec{AC}$. *[4 marks]*

(b) Hence find the area of the triangle ABC. *[2 marks]*

3. *[Maximum mark: 5]*

Given $\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 4 & -3 \\ 1 & 1 \end{pmatrix}$, find the matrix $\boldsymbol{X}$, such that $\boldsymbol{AXA}^{-1} = \boldsymbol{B}$.

4. *[Maximum mark: 5]*

The probability density function of the random variable X is defined as

$$f(x) = \begin{cases} \sin x, & 0 \le x \le \frac{\pi}{2} \\ 0, & \text{otherwise.} \end{cases}$$

Find $\mathrm{E}(X)$.

5. *[Maximum mark: 6]*

Paint is poured into a tray where it forms a circular pool with a uniform thickness of 0.5 cm. If the paint is poured at a constant rate of $4\,\text{cm}^3\text{s}^{-1}$, find the rate of increase of the radius of the circle when the radius is 20 cm.

6. *[Maximum mark: 6]*

The matrix $\boldsymbol{A}$ is such that $\boldsymbol{A}^2 = \boldsymbol{I}$, where $\boldsymbol{I}$ is the identity matrix. Use mathematical induction to prove that $(\boldsymbol{A}+\boldsymbol{I})^n = 2^{n-1}(\boldsymbol{A}+\boldsymbol{I})$, for all $n \in \mathbb{Z}^+$.

7. *[Maximum mark: 7]*

A curve is defined by the equation $8y\ln x - 2x^2 + 4y^2 = 7$. Find the equation of the tangent to the curve at the point where $x = 1$ and $y > 0$.

8. *[Maximum mark: 6]*

The first terms of an arithmetic sequence are $\frac{1}{\log_2 x}, \frac{1}{\log_8 x}, \frac{1}{\log_{32} x}, \frac{1}{\log_{128} x}, \ldots$

Find x if the sum of the first 20 terms of the sequence is equal to 100.

9. *[Maximum mark: 6]*

Two events A and B are such that $\mathrm{P}(A \cup B) = 0.7$ and $\mathrm{P}(A \mid B') = 0.6$.

Find $\mathrm{P}(B)$.

10. *[Maximum mark: 7]*

(a) Find all values of x for $0.1 \le x \le 1$ such that $\sin(\pi x^{-1}) = 0$. *[2 marks]*

(b) Find $\int_{\frac{1}{n+1}}^{\frac{1}{n}} \pi x^{-2} \sin(\pi x^{-1})\,\mathrm{d}x$, showing that it takes different integer values when n is even and when n is odd. *[3 marks]*

(c) Evaluate $\int_{0.1}^{1} \left| \pi x^{-2} \sin(\pi x^{-1}) \right| \mathrm{d}x$. *[2 marks]*

*Do **NOT** write solutions on this page.*

SECTION B

*Answer **all** questions in the answer booklet provided. Please start each question on a new page.*

11. *[Maximum mark: 19]*

(a) (i) Express $\cos\left(\frac{\pi}{6}+x\right)$ in the form $a\cos x - b\sin x$ where $a,\, b \in \mathbb{R}$.

(ii) Hence solve $\sqrt{3}\cos x - \sin x = 1$ for $0 \le x \le 2\pi$. *[7 marks]*

(b) Let $p(x) = 2x^3 - x^2 - 2x + 1$.

(i) Show that $x = 1$ is a zero of p.

(ii) Hence find all the solutions of $2x^3 - x^2 - 2x + 1 = 0$.

(iii) Express $\sin 2\theta\cos\theta + \sin^2\theta$ in terms of $\sin\theta$.

(iv) Hence solve $\sin 2\theta\cos\theta + \sin^2\theta = 1$ for $0 \le \theta \le 2\pi$. *[12 marks]*

*Do **NOT** write solutions on this page.*

12. *[Maximum mark: 19]*

(a) Express $4x^2 - 4x + 5$ in the form $a(x-h)^2 + k$ where $a, h, k \in \mathbb{Q}$. *[2 marks]*

(b) The graph of $y = x^2$ is transformed onto the graph of $y = 4x^2 - 4x + 5$. Describe a sequence of transformations that does this, making the order of transformations clear. *[3 marks]*

The function f is defined by $f(x) = \dfrac{1}{4x^2 - 4x + 5}$.

(c) Sketch the graph of $y = f(x)$. *[2 marks]*

(d) Find the range of f. *[2 marks]*

(e) By using a suitable substitution show that $\int f(x)\,\mathrm{d}x = \dfrac{1}{4}\int \dfrac{1}{u^2 + 1}\,\mathrm{d}u$. *[3 marks]*

(f) Prove that $\displaystyle\int_1^{3.5} \frac{1}{4x^2 - 4x + 5}\,\mathrm{d}x = \frac{\pi}{16}$. *[7 marks]*

*Do **NOT** write solutions on this page.*

13. *[Maximum mark: 22]*

On Saturday, Alfred and Beatrice play 6 different games against each other. In each game, one of the two wins. The probability that Alfred wins any one of these games is $\frac{2}{3}$.

(a) Show that the probability that Alfred wins exactly 4 of the games is $\frac{80}{243}$. *[3 marks]*

(b) (i) Explain why the total number of possible outcomes for the results of the 6 games is 64.

(ii) By expanding $(1+x)^6$ and choosing a suitable value for x, prove

$$64 = \binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6}.$$

(iii) State the meaning of this equality in the context of the 6 games played. *[4 marks]*

(c) The following day Alfred and Beatrice play the 6 games again. Assume that the probability that Alfred wins any one of these games is still $\frac{2}{3}$.

(i) Find an expression for the probability Alfred wins 4 games on the first day and 2 on the second day. Give your answer in the form $\binom{6}{r}^2 \left(\frac{2}{3}\right)^s \left(\frac{1}{3}\right)^t$ where the values of r, s and t are to be found.

(ii) Using your answer to (c)(i) and 6 similar expressions write down the probability that Alfred wins a total of 6 games over the two days as the sum of 7 probabilities.

(iii) Hence prove that $\binom{12}{6} = \binom{6}{0}^2 + \binom{6}{1}^2 + \binom{6}{2}^2 + \binom{6}{3}^2 + \binom{6}{4}^2 + \binom{6}{5}^2 + \binom{6}{6}^2$. *[9 marks]*

(d) Alfred and Beatrice play n games. Let A denote the number of games Alfred wins. The expected value of A can be written as $\mathrm{E}(A) = \sum_{r=0}^{n} r \binom{n}{r} \frac{a^r}{b^n}$.

(i) Find the values of a and b.

(ii) By differentiating the expansion of $(1+x)^n$, prove that the expected number of games Alfred wins is $\frac{2n}{3}$. *[6 marks]*

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